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Quantum phases of supersymmetric lattice models

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We review recent results on lattice models for spin-less fermions with strong repulsive interactions. A judicious tuning of kinetic and interaction terms leads to a model possessing supersymmetry. In the 1D case, this model displays critical behavior described by superconformal field theory. On 2D lattices we generically find superfrustration, characterized by an extensive ground state entropy. For certain 2D lattices analytical results on the ground state structure reveal yet another quantum phase, which we tentatively call 'supertopological'.

Keywords: Strongly correlated lattice fermions, supersymmetry, topological phases

1. Introduction

A long standing challenge in condensed matter physics is to understand the properties of strongly correlated electron systems. While it is relatively easy to formulate model descriptions, it has proved exceedingly difficult to arrive at exact results for these in spatial dimensions $D > 1$, in particular in regimes where interaction and kinetic terms in the relevant hamiltonian are of comparable strength. In a series of papers, initiated by one of the authors together with P. Fendley and J. de Boer,[1] a specific model for lattice fermions was explored, precisely in this non-perturbative regime. The key property of this model is **supersymmetry**. This gives a subtle balance between kinetic and interaction terms, leading to remarkable features such as, in particular, large degeneracies of quantum ground states. At the same time, supersymmetry provides a rich mathematical structure that can be employed to derive rigorous results for some of the key features of the model.

The degrees of freedom are spin-less fermionic particles living on a given lattice. A fermion at site i is created by the operator $c_i^\dagger$ with $\{c_i, c_j^\dagger\} = \delta_{ij}$. The fermions have a *hard core*, meaning that the presence of a fermion on a given site excludes the occupation of all adjacent sites. With this, the fermion creation operator becomes $d_i^\dagger = c_i^\dagger \mathcal{P}_{<i>}$, where

$$\mathcal{P}_{<i>} = \prod_{j \text{ next to } i} (1 - c_j^\dagger c_j) \tag{1}$$

is zero if any site next to i is occupied. The hamiltonian is defined in terms of the

supercharge $Q = \sum_i d_i^\dagger$:

$$H = \{Q, Q^\dagger\} = \sum_{<ij>} d_i^\dagger d_j + \sum_i \mathcal{P}_{<i>}. \tag{2}$$

The model is supersymmetric because $Q^2 = (Q^\dagger)^2 = 0$, which then implies that $[Q, H] = [Q^\dagger, H] = 0$. The second term in the hamiltonian combines a chemical potential and a repulsive next-nearest-neighbor potential. The details of this term depend on the lattice we choose.

A powerful analytical tool is the Witten index

$$W = \mathrm{Tr}[(-1)^F] \tag{3}$$

with F the operator counting the number of fermions in a given state. One easily shows that $W = N_b - N_f$, where N_b is the number of bosonic ground states (those with an even number of fermions), and N_f is the number of fermionic ground states (those with an odd number of fermions).[2] By this result, $|W|$ is a lower bound on the number of ground states. In analyzing the supersymmetric model we also used techniques from cohomnology theory, relying on a one-to-one correspondence between quantum ground states and cohomology classes of the complex associated with the supercharge Q.

The papers[1,3–6] explored this model on a variety of lattices and revealed a number of remarkable quantum phases. While some of the specifics of these phases require the fine-tuning set by supersymmetry, many qualitative features are expected to survive in models that are sufficiently close to the supersymmetric point.

On 1D lattices (quantum chains), the model turns out to be quantum critical, with the critical behavior fully described by the first ($k = 1$) unitary minimal model of $N = 2$ supersymmetric Conformal Field Theory (SCFT).[1]

On generic 2D and 3D lattices, the supersymmetric lattice model displays a phenomenon we call **superfrustration**. This term denotes a strong form of quantum charge frustration, with the number of quantum ground states growing exponentially with the volume of the system, implying extensive ground state entropy.[4,5] For an elaborate account we refer the reader to an earlier review.[7]

In this paper we focus on 2D lattices which exhibit yet another type of anomalous behavior, with the number of quantum ground states growing exponentially with the *linear* dimensions of the system (sub-extensive ground state entropy). One example is the 2D square lattice, where this property is established via a rigorous theorem relating the number of quantum ground states to specific rhombus tilings of the plane. A second example is the octagon-square lattice. In this latter case the ground state structure is less involved, allowing us to delve a bit deeper and to extend the analysis to the presence of defects. With some of the features observed (torus degeneracies, presence of edge modes) being reminiscent of those of topological phases of 2D matter, we tentatively refer to these phases as 'supertopological'.

Table 1. Witten Index for $M \times N$ square lattice.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	-1	1	3	1	-1	1	3	1	-1	1	3	1	-1	1
3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4
4	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1
5	1	1	1	1	-9	1	1	1	1	11	1	1	1	1	-9
6	1	-1	4	3	1	14	1	3	4	-1	1	18	1	-1	4
7	1	1	1	1	1	1	1	1	1	1	1	1	1	-27	1
8	1	3	1	7	1	3	1	7	1	43	1	7	1	3	1
9	1	1	4	1	1	4	1	1	40	1	1	4	1	1	4
10	1	-1	1	3	11	-1	1	43	1	9	1	3	1	69	11
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166	1	3	4

2. Square lattice

Where numerical studies of the Witten index[4,5] showed superfrustration for a wide range of 2D lattices, they revealed a very different behavior for the square lattice wrapped around the torus (see Table 1). At first glance one notices that the index does not grow exponentially with the system size as it does for the superfrustrated systems. Inspired by the curiosities of this table and two conjectures[8] on its structure, Jonsson[9] proved a general expression for the Witten index $W_{u,v}$ of the square lattice with periodic boundary conditions given by the vectors $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. He showed that $W_{u,v}$ is simply related to tilings constructed from the rhombuses pictured in Fig. 1. Precisely, let t_b (t_f) be the number of ways of tiling the torus with these four rhombus types, so that there are an even (odd) number of rhombuses.

Theorem 2.1 (Jonsson, 2006). *The expression for the Witten index reads*

$$W_{u,v} = N_b - N_f = t_b - t_f - (-1)^d \theta_d \theta_{d^*}, \tag{4}$$

where $d \equiv gcd(u_1 - u_2, v_1 - v_2)$, $d^* \equiv gcd(u_1 + u_2, v_1 + v_2)$ *and*

$$\theta_d \equiv \begin{cases} 2 & \textit{if } d = 3k, \textit{ with } k \textit{ integer} \\ -1 & \textit{otherwise.} \end{cases} \tag{5}$$

A natural extension of Jonsson's theorem is to relate the *total* number of ground states to rhombus tilings. Exploiting the one-to-one correspondence between ground states and elements of the cohomology of the supercharge Q, we were able to prove this relation explicitly when $\vec{u} = (m, -m)$ and $v_1 + v_2 = 3p$.[6,10]

Theorem 2.2 (Fendley, Huijse and Schoutens, 2009). *The total number of ground states reads*

$$N_b + N_f = t_b + t_f + \Delta, \tag{6}$$

where $|\Delta| = |\theta_d \theta_{d^*}|$. *For* $\vec{u} = (m, -m)$ *and* $v_1 + v_2 = 3p$ *we find* $\Delta = -(-1)^{(\theta_m + 1)p} \theta_d \theta_{d^*}$.

In this correspondence, the number of fermions in a given ground state matches the number of tiles in the corresponding tiling. Although the proof is restricted to

a certain set of periodicities, there is strong evidence that the theorem holds for general $\vec{u}$ and $\vec{v}$. We explicitly checked the result for a variety of small systems.

The relation between tilings and ground states implies that for large enough systems ground states exist at all rational fillings (particles per site) F/N between $1/5$ and $1/4$ (see also[11] for an alternative proof). One can show that for $\vec{v} = (n, n)$ and $m = 3p$, $n = 3q$, the number of tilings grows as $4^{p+q}/\sqrt{pq}$, thus establishing sub-extensive ground state entropy.[12]

For free boundary conditions in either one or both of the diagonal directions along the square lattice ($(m, -m)$ and (n, n)) the number of ground states reduces dramatically.[6,13] One finds that it is either one or zero, except for the cylindrical case periodic in the $(m, -m)$-direction with $m = 3p$ and $n = 3q + 2$ or $n = 3q + 3$. In that case the number of ground states is $4^{(q+1)}$.

With the above results in place, the ground state counting problem for the square lattice has been completely settled. Further pressing questions concern the nature of these ground states and of the excited state spectrum. Some progress on these matters was provided in,[6] where we presented numerical results strongly indicating the presence of critical modes in ladder versions of the 2D lattice. We then argued that the full 2D lattice with (diagonal) open boundary conditions supports edge modes described by $N = 2$ superconformal field theory.

While the physical understanding of the quantum phase on the square lattice remains far from complete, one is led to a picture where the ground state corresponding to a given tiling has fermions that are confined to the area set by an individual tile, but quantum fluctuating within that space. For the particular case of a ladder with periodicity vectors $(1, 2)$ and $(L, 0)$, closed form expressions for the ground states at filling $1/4$ confirm this picture; for the more general case such explicit expressions are not available. The tiling based physical picture of the ground state wavefunctions is reminiscent of electrons in a filled magnetic Landau level, each of them effectively occupying an area set by the strength of the magnetic field. Critical edge modes naturally fit into a picture of this sort.

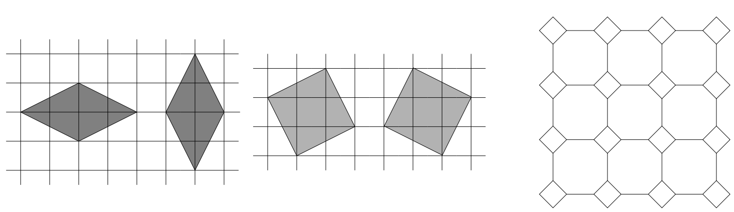

Fig. 1. On the left the four rhombuses on the square lattice, on the right the octagon-square lattice of size 4×4.

3. Octagon-square lattice

A second 2D lattice where the supersymmetric model displays sub-extensive ground state entropy is the octagon-square lattice (Fig. 1). The growth behavior of the numbers of ground states on the plane, cylinder and torus is similar to that of the square lattice. A big difference, however, is that here all ground states reside at 1/4 filling. This hugely simplifies the computation of the degeneracies. For the plane we find that the ground state is unique. For the cylinder with $M \times L$ square plaquettes, where M is the number of square plaquettes along the periodic, horizontal direction and L along the open, vertical direction, the number of ground states is 2^L. Finally, for the torus the number of ground states is $2^M + 2^L - 1$.[4]

There is again a relatively simple physical picture which we propose as a basis for further analysis of physical properties. This picture reflects the systematics uncovered by the analysis of the associated cohomology problem as well as results for small system sizes. The basic building block of the many-body ground states is the 1-fermion ground state on an isolated square plaquette. The unique many-body ground state on the plane essentially has individual fermions occupying this lowest 1-plaquette orbital, again allowing the analogy with a filled Landau level. Closing boundaries leads to the possibility that electrons on horizontal or vertical rows of plaquettes 'shift' into a second 1-fermion state, this way building up the total of $2^M + 2^L - 1$ ground states. This picture can be further substantiated by allowing defects, which we bring in by adding diagonal connections in individual plaquettes.

Among the key issues that are presently on the agenda for further study are: the existence of energy gaps, the presence of bulk or edge critical modes, and interactions and braiding properties of defects. We are confident that the constraints set by the supersymmetry, which have been instrumental in the progress made so far, will allow further progress in the analysis of the remarkable 'supertopological' phases on the square and octagon-square lattices.

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